NON-MINIMAL PP-WAVE EINSTEIN-YANG-MILLS-HIGGS MODEL: COLOR CROSS-EFFECTS INDUCED BY CURVATURE

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Abstract

Non-minimal interactions in the pp-wave Einstein - Yang - Mills - Higgs (EYMH) model are shown to give rise to color cross-effects analogous to the magneto-electricity in the Maxwell theory. In order to illustrate the significance of these color cross-effects, we reconstruct the effective (associated, color and color-acoustic) metrics for the pp-wave non-minimal seven-parameter EYMH model with parallel gauge and scalar background fields. Then these metrics are used as hints for obtaining explicit exact solutions of the non-minimally extended Yang-Mills and Higgs equations for the test fields propagating in the vacuum interacting with curvature. The influence of the non-minimal coupling on the test particle motion is interpreted in terms of the so-called trapped surfaces, introduced in the Analog Gravity theory.

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1 Introduction

Magneto-electricity is the well-known linear cross-effect appearing in (moving) anisotropic media [1, 2]. The interest to this problem has been recently revived due to discussions about the Feigel effect (see [3, 4, 5, 6] and references therein), which takes place in crossed external electric and magnetic field. In this connection we have to stress that the non-minimal interactions (induced by curvature) also produce magneto-electric effect in the $U(1)$ symmetric electrodynamic systems and color cross-effects in the $SU(N)$ symmetric Yang-Mills-Higgs systems. The signal about cross-effects appear when we analyze the linear response tensor in the non-minimal Einstein-Maxwell theory (see, e.g., [7, 8, 9]) and non-minimal Einstein-Yang-Mills-Higgs theory (see, e.g., [10]). When a medium possesses cross-effects the theory of wave propagation becomes much more sophisticated. In order to clarify basic properties of such processes, a formalism of effective metrics is being widely used (see, e.g., [2, 11, 12, 13]). The development of this formalism exhibits few interesting details. When we deal with propagation of electromagnetic waves in vacuum, the inverse spacetime metric g^{ik} can be regarded both: as an *optical* metric and as an *associated* metric. The first term means, that the photon with the momentum four-vector p_k moves in pure vacuum along the null geodesic line of this spacetime, and the eikonal equation $g^{ik}p_ip_k = 0$ is satisfied. The second term, associated metric, means that the linear constitutive equation $H^{ik} = C^{ikmn}F_{mn}$, linking the Maxwell tensor F_{mn}, the strength of the electromagnetic field, with the induction tensor H^{ik}[14, 2], can be reconstructed for the vacuum case if we put $C^{ikmn} = 1/2(g^{im}g^{kn} - g^{in}g^{km})$, i.e., using the quadratic combinations of the inverse metric g^{ik}. For the spatially isotropic medium Gordon [15] introduced the optical metric $g^{*ik} = g^{ik}+(n^2-1)U^iU^k$ (we use the rule $U^kU_k = 1$ for the normalization of the velocity four-vector U^k). The propagation of photons in such spatially isotropic medium with a refraction index n is equivalent to the motion in the effective spacetime with metric g^{*ik}, the eikonal equation $g^{*ik}p_ip_k = 0$ being valid. On the other hand, the tensor of linear response C^{ikmn} can be reconstructed as one for the vacuum, if to replace g^{ik} by the g^{*ik} [16, 14]. Thus, the metric g^{*ik} is both optical and associated one for the spatially isotropic medium.

When the medium is anisotropic, but possesses uniaxial symmetry and cross magneto-electric effects are absent, then the quartic Fresnel surface [2] is factorized into the product of two light cones, and *two* optical metrics of the Lorentzian type are sufficient to describe the photon propagation [17]. These optical metrics can be constructed by using not only the terms g^{ik} and U^iU^k, but X^iX^k also, where X^k is a space-like four-vector pointing the direction of anisotropy in the space. Such bimetricity relates to the birefringence properties of the anisotropic media. In its turn, the tensor of linear response C^{ikmn} for the uniaxial medium without cross-effects can also be reconstructed in terms of quadratic combinations of two optical metrics [2, 18], but the corresponding decomposition is more complicated than the one for the spatially isotropic medium. When the anisotropic medium is biaxial, Perlick [17] assumes that the Lorentzian optical metrics do not exist, and one need to use the optical metrics of the Finsler type, i.e., depending not only on the coordinates, but on the particle four-vector of momentum also.

When a medium possesses cross magneto-electric effects, the quartic Fresnel surface can not be factorized, in general, into the product of two light cones ([4, 6]). However it becomes possible, when the magneto-electric coefficients obey some special requirements, and the authors of [6] gave an example of such explicit reconstruction of the two optical metrics. In order to find associated metrics for the medium, possessing cross magneto-electric effects, we focus on the formalism proposed in [18]. This generalized formalism is based on the introduction of a new effective space, in which the symmetric tensor fields of the rank two (the associated metrics), which we use for the reconstruction of the tensor C^{ikmn}, are considered as vectors. When C^{ikmn} is symmetric with respect to the transposition of the pairs of indices ik and mn, this effective space is four-dimensional, since arbitrary linear response tensor can always be decomposed into the sum of four linearly independent terms $X^i_{(a)}X^k_{(a)}$ $((a) = (0), (1), (2), (3))$, where $X^k_{(a)}$ are tetrad four-vectors. The representation of the linear response tensor by the associated metrics is not unique, thus, we considered in [18] transformations in this effective space related to the transition from one set of associated metrics to another one. Based on this idea, we can conclude, that when we deal with vacuum and spatially isotropic medium, it is sufficient to use one-dimensional sub-space of this effective space, since there exists a unique optical metric, reconstructing the tensor C^{ikmn}. In the uniaxial medium without cross-effects it is sufficient to use two-dimensional sub-space, since two optical metrics reconstruct the constitutive equations. As for the biaxial case we need three or four associated metrics, thus, the number of optical metrics (again two, since the photons have two degrees of freedom) does not coincide with the number of the associated ones.

In order to illustrate this idea, we consider here the associated metrics and color metrics (generalization of the optical metrics for the Yang-Mills field) for the pp-wave non-minimal Einstein-Yang-Mills-Higgs (EYMH) model. The symmetry of this model is not uniaxial, the cross-effect is present, nevertheless, color metrics and linear response tensor can be reconstructed explicitly.

The paper is organized as follows. In Section 2 we introduce briefly the action functional and master equations for the gauge, scalar and gravitational fields in the framework of seven-parameter non-minimal EYMH model. In Section 3 we apply this non-minimal EYMH model to the field configuration with pp-wave symmetry and reconstruct, subsequently, associated, color and color-acoustic metrics appearing in the pp-wave non-minimal EYMH model. In Sections 4 and 5, using the obtained color and color-acoustic metrics as hints, we obtain the exact parallel in the group space solutions to the Yang-Mills and Higgs equations, respectively. In Discussion we interpret the obtained results in terms of trapped surfaces, introduced in the theory of Analog Gravity.

2 Seven-parameter non-minimal EYMH model

2.1 Basic definitions

Consider an action functional

$$S_{(\mathrm{NMEYMH})} = \int d^4x\sqrt{-g}\ \left\{\frac{R+2\Lambda}{\kappa} + \frac{1}{2}F^{(a)}_{ik}F^{ik}_{(a)} - D_m\Phi^{(a)}D^m\Phi_{(a)} + m^2\Phi^2 + \right.$$
$$\left. + \frac{1}{2}\mathcal{R}^{ikmn}X_{(a)(b)}F^{(a)}_{ik}F^{(b)}_{mn} - \Re^{mn}Y_{(a)(b)}\hat{D}_m\Phi^{(a)}\hat{D}_n\Phi^{(b)}\right\}, \quad (1)$$

where the so-called susceptibility tensors $\mathcal{R}^{ikmn}$ and $\Re^{mn}$ are defined as follows:

$$\mathcal{R}^{ikmn} \equiv \frac{q_1}{2}R\left(g^{im}g^{kn}-g^{in}g^{km}\right)+\frac{q_2}{2}(R^{im}g^{kn}-R^{in}g^{km}+R^{kn}g^{im}-R^{km}g^{in})+q_3R^{ikmn}, \quad (2)$$

$$\Re^{mn} \equiv q_4Rg^{mn} + q_5R^{mn}. \quad (3)$$

Here $g = \det(g_{ik})$ is the determinant of a metric tensor g_{ik}, R is the Ricci scalar, R_{mn} is the Ricci tensor, $R^i{}_{klm}$ is the Riemann tensor. Latin indices run from 0 to 3, the group index (a) runs from 1 to N^2-1. $F^{(a)}_{mn}$ is the tensor of the Yang-Mills real field strength (see, e.g., [19, 20])

$$F^{(a)}_{mn}=\nabla_mA^{(a)}_n-\nabla_nA^{(a)}_m + \mathcal{G}f^{(a)}_{\ (b)(c)}A^{(b)}_mA^{(c)}_n, \quad (4)$$

where $A^{(a)}_i$ is the Yang-Mills field potential four-vector, ∇_k is the covariant derivative. The symbol $\Phi^{(a)}$ denotes the multiplet of real scalar fields ($SU(N)$ symmetric Higgs fields). The gauge covariant derivative $\hat{D}_m\Phi^{(a)}$ is defined as [19]

$$\hat{D}_m\Phi^{(a)} \equiv \nabla_m\Phi^{(a)} + \mathcal{G}f^{(a)}_{\ (b)(c)}A^{(b)}_m\Phi^{(c)}. \quad (5)$$

m is a mass the Higgs field, and q_1, q_2, ... q_5 are the constants of non-minimal coupling. In the definitions we follow the book [19], i.e., we use the following basic formulas:

$$G_{(a)(b)} \equiv 2\mathrm{Tr}\ \mathbf{t}_{(a)}\mathbf{t}_{(b)}, \quad \left[\mathbf{t}_{(a)},\mathbf{t}_{(b)}\right] = if^{(c)}_{\ (a)(b)}\mathbf{t}_{(c)}, \quad (6)$$

$$f_{(c)(a)(b)} \equiv G_{(c)(d)}f^{(d)}_{\ (a)(b)} = -2i\ \mathrm{Tr}\ \left[\mathbf{t}_{(a)},\mathbf{t}_{(b)}\right]\mathbf{t}_{(c)}, \quad (7)$$

$$\mathbf{F}_{mn} = -i\mathcal{G}\mathbf{t}_{(a)}F^{(a)}_{mn}, \quad \mathbf{A}_m = -i\mathcal{G}\mathbf{t}_{(a)}A^{(a)}_m, \quad \mathbf{\Phi} = \mathbf{t}_{(a)}\Phi^{(a)}. \quad (8)$$

The symmetric tensor $G_{(a)(b)}$ plays a role of a metric in the group space, $\mathbf{t}_{(a)}$ are the Hermitian traceless generators of $SU(N)$ group, $f^{(d)}_{\ (a)(b)}$ are the structure constants of the $SU(N)$ group, the constant $\mathcal{G}$ is the strength of the gauge coupling. The quantities $X_{(a)(b)}$ and $Y_{(a)(b)}$ have the following structure:

$$X_{(a)(b)} \equiv G_{(a)(b)} + (Q_1-1)\frac{\Phi_{(a)}\Phi_{(b)}}{\Phi^2}, \quad Y_{(a)(b)} \equiv G_{(a)(b)} + (Q_2-1)\frac{\Phi_{(a)}\Phi_{(b)}}{\Phi^2}, \quad (9)$$

with two new coupling constants Q_1 and Q_2. When $Q_1 = Q_2 = 1$, the tensors $X_{(a)(b)}$ and $Y_{(a)(b)}$ coincide with the metric of the group space. When $Q_1 = 0$, $X_{(a)(b)} = P_{(a)(b)}$, and when $Q_2 = 0$, $Y_{(a)(b)} = P_{(a)(b)}$, where

$$P_{(a)(b)} \equiv G_{(a)(b)} - \frac{\Phi_{(a)}\Phi_{(b)}}{\Phi^2}, \quad (10)$$

is a projector in the group space, which possesses the properties:

$$P_{(a)(b)} = P_{(b)(a)}\,, \quad P_{(a)(b)}\Phi^{(b)} = 0\,, \quad P_{(a)(b)}P^{(a)(c)} = P^{(c)}_{(b)}\,, \quad P^{(a)}_{(a)} = N^2 - 2\,. \tag{11}$$

We assume here that the multiplet $\Phi^{(a)}$ has positive norm $G_{(a)(b)}\Phi^{(a)}\Phi^{(b)} \equiv \Phi^2 > 0$, allowing us to equip the group space by the vector $q^{(a)} = \Phi^{(a)}/\Phi$ with the unit norm $G_{(a)(b)}q^{(a)}q^{(b)} = 1$. Finally, the tensor $F^{ik}_{(a)}$ satisfies the relation

$$\hat{D}_k F^{*ik}_{(a)} = 0\,, \quad F^{*ik}_{(a)} = \frac{1}{2}\epsilon^{ikls}F_{ls(a)}\,, \tag{12}$$

where $\epsilon^{ikls} = \frac{1}{\sqrt{-g}}E^{ikls}$ is the Levi-Civita tensor, E^{ikls} is the completely skew - symmetric symbol with $E^{0123} = -E_{0123} = 1$.

2.2 Non-minimal extension of the Yang-Mills equations

The variation of the action functional over the Yang-Mills potential $A^{(a)}_i$ yields

$$\hat{D}_k \mathcal{H}^{ik}_{(a)} = \mathcal{G}(\hat{D}_k\Phi^{(d)})f^{(b)}_{\ (a)(h)}\Phi^{(h)}\left[G_{(b)(d)}g^{ik} + Y_{(b)(d)}\Re^{ik}\right], \tag{13}$$

where the tensor $\mathcal{H}^{ik}_{(a)}$ is

$$\mathcal{H}^{ik}_{(a)} = F^{ik}_{(a)} + \mathcal{R}^{ikmn}X_{(a)(b)}F^{(b)}_{mn} \equiv C^{ikmn}_{(a)(b)}F^{(b)}_{mn}\,. \tag{14}$$

Thus, the linear response tensor $C^{ikmn}_{(a)(b)}$, linking the Yang-Mills field strength $F^{(b)}_{mn}$ and the so-called color induction tensor $\mathcal{H}^{ik}_{(a)}$, is of the form

$$C^{ikmn}_{(a)(b)} \equiv \left[\frac{1}{2}(g^{im}g^{kn} - g^{in}g^{km}) + \mathcal{R}^{ikmn}\right]G_{(a)(b)} + (Q_1 - 1)\frac{\Phi_{(a)}\Phi_{(b)}}{\Phi^2}\mathcal{R}^{ikmn}\,. \tag{15}$$

The definitions of the tensors of color permittivity, $\varepsilon^{im}_{(a)(b)}$, color impermeability, $(\mu^{-1})^{pq}_{(a)(b)}$ and color cross-effects, $\nu^{pm}_{(a)(b)}$, are the following

$$\varepsilon^{im}_{(a)(b)}{=}2\mathcal{C}^{ikmn}_{(a)(b)}U_kU_n\,, \quad (\mu^{-1})^{pq}_{(a)(b)}{=}{-}\frac{1}{2}\eta^{p}_{\ ik}\mathcal{C}^{ikmn}_{(a)(b)}\eta_{mn}^{\ \ \ q}\,, \quad \nu^{pm}_{(a)(b)}{=}\eta^{p}_{\ ik}\mathcal{C}^{ikmn}_{(a)(b)}U_n\,, \tag{16}$$

where $\eta^{ikm} \equiv \epsilon^{ikmn}U_n$. Using (15) and the standard definition of the projector, $\Delta^{ik} \equiv g^{ik} - U^iU^k$, we obtain explicitly

$$\varepsilon^{im}_{(a)(b)} = \left[\Delta^{im} + 2\mathcal{R}^{ikmn}U_kU_n\right]G_{(a)(b)} + 2(Q_1 - 1)\mathcal{R}^{ikmn}U_kU_n\frac{\Phi_{(a)}\Phi_{(b)}}{\Phi^2}\,, \tag{17}$$

$$(\mu^{-1})^{pq}_{(a)(b)} = \left[\Delta^{pq} - 2\ {}^*\mathcal{R}^{*plqs}U_lU_s\right]G_{(a)(b)} - 2(Q_1 - 1)\ {}^*\mathcal{R}^{*plqs}U_lU_s\frac{\Phi_{(a)}\Phi_{(b)}}{\Phi^2}\,, \tag{18}$$

$$\nu^{pm}_{(a)(b)} = -{}^*\mathcal{R}^{plnm}U_lU_n\left[G_{(a)(b)} + (Q_1 - 1)\frac{\Phi_{(a)}\Phi_{(b)}}{\Phi^2}\right]. \tag{19}$$

Assuming that there is only one preferred direction in the group space and this direction is associated with the vector $q^{(a)}$, one can decompose $C^{ikmn}_{(a)(b)}$ as

$$C^{ikmn}_{(a)(b)} = q_{(a)}q_{(b)}C^{ikmn}_{(\mathrm{long})} + P_{(a)(b)}C^{ikmn}_{(\mathrm{trans})}\,, \tag{20}$$

where the longitudinal and transversal parts are, respectively

$$C^{ikmn}_{(\rm long)} \equiv C^{ikmn}_{(a)(b)} q^{(a)} q^{(b)} = \left[\frac{1}{2}(g^{im}g^{kn} - g^{in}g^{km}) + Q_1 \mathcal{R}^{ikmn}\right] , \tag{21}$$

$$C^{ikmn}_{(\rm trans)} \equiv \frac{1}{(N^2-2)} C^{ikmn}_{(a)(b)} P^{(a)(b)} = \left[\frac{1}{2}(g^{im}g^{kn} - g^{in}g^{km}) + \mathcal{R}^{ikmn}\right] . \tag{22}$$

Mention that $C^{ikmn}_{(\rm trans)}$ can be obtained from $C^{ikmn}_{(\rm long)}$ by the formal replacement $Q_1 \to 1$, and let us use this features below for the simplifications of the formulas. The corresponding longitudinal and transversal components of $\varepsilon^{im}_{(a)(b)}$, $(\mu^{-1})^{pq}_{(a)(b)}$ and $\nu^{pm}_{(a)(b)}$ can be easily written in analogy with (21) and (22).

2.3 Non-minimal extension of the Higgs field equations

The variation of the action $S_{(\rm NMEYMH)}$ over the Higgs scalar field $\Phi^{(a)}$ yields

$$\hat{D}_m \left\{ \left[g^{mn} G_{(a)(b)} + \Re^{mn} Y_{(a)(b)} \right] \hat{D}_n \Phi^{(b)} \right\} =$$

$$-m^2 \Phi_{(a)} - \frac{(Q_1{-}1)}{2\Phi^2} \; \mathcal{R}^{ikmn} F^{(c)}_{ik} F^{(b)}_{mn} \Phi_{(b)} \left[G_{(a)(c)} - \frac{\Phi_{(a)}\Phi_{(c)}}{\Phi^2} \right] +$$

$$+ \frac{(Q_2{-}1)}{\Phi^2} \; \Re^{mn} \left[G_{(a)(c)} - \frac{\Phi_{(a)}\Phi_{(c)}}{\Phi^2} \right] \Phi_{(b)} (\hat{D}_m \Phi^{(c)}) (\hat{D}_n \Phi^{(b)}) \, . \tag{23}$$

This equation, clearly, has a form

$$\hat{D}_m \left[\mathcal{C}^{mn}_{(a)(b)} \hat{D}_n \Phi^{(b)} \right] = \mathcal{I}_{(a)} \, , \tag{24}$$

where the tensor $\mathcal{C}^{mn}_{(a)(b)}$ is

$$\mathcal{C}^{mn}_{(a)(b)} = \left[g^{mn} + \Re^{mn} \right] G_{(a)(b)} + (Q_2 - 1) \Re^{mn} \frac{\Phi_{(a)}\Phi_{(b)}}{\Phi^2} \, , \tag{25}$$

and $\mathcal{I}_{(a)}$ stands for the right-hand-side of (23). $\mathcal{C}^{mn}_{(a)(b)}$ can also be decomposed into longitudinal and transversal components

$$\mathcal{C}^{ik}_{(a)(b)} = \mathcal{C}^{ik}_{(\rm long)} q_{(a)} q_{(b)} + P_{(a)(b)} \mathcal{C}^{ik}_{(\rm trans)} \, , \tag{26}$$

where

$$\mathcal{C}^{ik}_{(\rm long)} \equiv \mathcal{C}^{ik}_{(a)(b)} q^{(a)} q^{(b)} \, , \quad \mathcal{C}^{ik}_{(\rm trans)} \equiv \frac{1}{(N^2-2)} \mathcal{C}^{ik}_{(a)(b)} P^{(a)(b)} \, . \tag{27}$$

The definitions

$$\tilde{g}^{ik}_{(\rm long)} \equiv \mathcal{C}^{ik}_{(\rm long)} = g^{ik} + Q_2 \Re^{ik} \, , \quad \tilde{g}^{ik}_{(\rm trans)} \equiv \mathcal{C}^{ik}_{(\rm trans)} = g^{ik} + \Re^{ik} \, , \tag{28}$$

introduce two color-acoustic metrics for the colored scalar particles. Again, $\tilde{g}^{ik}_{(\rm trans)}$ can be obtained from $\tilde{g}^{ik}_{(\rm long)}$ by the formal replacement $Q_2 \to 1$.

2.4 Master equations for the gravitational field

The equations for the gravity field related to the action functional $S_{(\mathrm{NMEYMH})}$ are of the form

$$R_{ik} - \frac{1}{2}R\ g_{ik} = \Lambda\ g_{ik} + \kappa \left[T^{(YM)}_{ik} + T^{(H)}_{ik} + T^{(NM)}_{ik}\right] . \qquad (29)$$

The term $T^{(YM)}_{ik}$:

$$T^{(YM)}_{ik} \equiv \frac{1}{4}g_{ik}F^{(a)}_{mn}F^{mn}_{(a)} - F^{(a)}_{in}F_{k(a)}^{\ \ n} , \qquad (30)$$

is a stress-energy tensor of the pure Yang-Mills field, the term $T^{(H)}_{ik}$:

$$T^{(H)}_{ik} \equiv -\frac{1}{2}g_{ik}\hat{D}_m\Phi^{(a)}\hat{D}^m\Phi_{(a)} + \hat{D}_i\Phi^{(a)}\hat{D}_k\Phi_{(a)} + \frac{1}{2}g_{ik}m^2\Phi^{(a)}\Phi_{(a)} \qquad (31)$$

is a stress-energy tensor for the scalar Higgs field. The non-minimal contributions enter the last tensor $T^{(NM)}_{ik}$, which may be represented as a sum of five items:

$$T^{(NM)}_{ik} \equiv q_1 T^{(I)}_{ik} + q_2 T^{(II)}_{ik} + q_3 T^{(III)}_{ik} + q_4 T^{(IV)}_{ik} + q_5 T^{(V)}_{ik} . \qquad (32)$$

The definitions of these five tensors relate to the corresponding coupling constant q_1,q_2, ...q_5. The tensors $T^{(I)}_{ik}$, $T^{(II)}_{ik}$, ... $T^{(V)}_{ik}$ are

$$T^{(I)}_{ik} = RX_{(a)(b)}\left[\frac{1}{4}g_{ik}F^{(a)}_{mn}F^{mn(b)} - F^{(a)}_{im}F_k^{\ m(b)}\right] - \frac{1}{2}R_{ik}X_{(a)(b)}F^{(a)}_{mn}F^{mn(b)} +$$
$$+ \frac{1}{2}\left[\hat{D}_i\hat{D}_k - g_{ik}\hat{D}^l\hat{D}_l\right]\left[X_{(a)(b)}F^{(a)}_{mn}F^{mn(b)}\right] , \qquad (33)$$

$$T^{(II)}_{ik} = -\frac{1}{2}g_{ik}\left[\hat{D}_m\hat{D}_l\left(X_{(a)(b)}F^{mn(a)}F^{l(b)}_{\ \ n}\right) - R_{lm}X_{(a)(b)}F^{mn(a)}F^{l(b)}_{\ \ n}\right] -$$
$$-F^{ln(a)}X_{(a)(b)}\left(R_{il}F^{(b)}_{kn} + R_{kl}F^{(b)}_{in}\right) - \frac{1}{2}\hat{D}^m\hat{D}_m\left(X_{(a)(b)}F^{(a)}_{in}F_k^{\ n(b)}\right) +$$
$$+\frac{1}{2}\hat{D}_l\left[\hat{D}_i\left(X_{(a)(b)}F^{(a)}_{kn}F^{ln(b)}\right) + \hat{D}_k\left(X_{(a)(b)}F^{(a)}_{in}F^{ln(b)}\right)\right] - R^{mn}X_{(a)(b)}F^{(a)}_{im}F^{(b)}_{kn} , \qquad (34)$$

$$T^{(III)}_{ik} = \frac{1}{4}g_{ik}R^{mnls}X_{(a)(b)}F^{(a)}_{mn}F^{(b)}_{ls} - \frac{3}{4}X_{(a)(b)}F^{ls(a)}\left(F_i^{\ n(b)}R_{knls} + F_k^{\ n(b)}R_{inls}\right) -$$
$$-\frac{1}{2}\hat{D}_m\hat{D}_n\left[X_{(a)(b)}\left(F_i^{\ n(a)}F_k^{\ m(b)} + F_k^{\ n(a)}F_i^{\ m(b)}\right)\right] , \qquad (35)$$

$$T^{(IV)}_{ik} = \left(R_{ik} - \frac{1}{2}g_{ik}R\right)Y_{(a)(b)}(\hat{D}_m\Phi^{(a)})(\hat{D}^m\Phi^{(b)}) + RY_{(a)(b)}(\hat{D}_i\Phi^{(a)})(\hat{D}_k\Phi^{(b)}) +$$
$$+\left(g_{ik}\hat{D}^n\hat{D}_n - \hat{D}_i\hat{D}_k\right)\left[Y_{(a)(b)}(\hat{D}_m\Phi^{(a)})(\hat{D}^m\Phi^{(b)})\right] , \qquad (36)$$

$$T^{(V)}_{ik} = Y_{(a)(b)}(\hat{D}_m\Phi^{(b)})\left[R^m_i(\hat{D}_k\Phi^{(a)}) + R^m_k(\hat{D}_i\Phi^{(a)})\right] - \frac{1}{2}R_{ik}Y_{(a)(b)}(\hat{D}_m\Phi^{(a)})(\hat{D}^m\Phi^{(b)}) -$$
$$-\frac{1}{2}\hat{D}^m\left\{\hat{D}_i\left[Y_{(a)(b)}(\hat{D}_m\Phi^{(a)})(\hat{D}_k\Phi^{(b)})\right] + \hat{D}_k\left[Y_{(a)(b)}(\hat{D}_m\Phi^{(a)})(\hat{D}_i\Phi^{(b)})\right] -\right.$$
$$\left.-\hat{D}_m\left[Y_{(a)(b)}(\hat{D}_i\Phi^{(a)})(\hat{D}_k\Phi^{(b)})\right]\right\} + \frac{1}{2}g_{ik}\hat{D}_m\hat{D}_n\left[Y_{(a)(b)}\left(\hat{D}^m\Phi^{(a)}\right)\left(\hat{D}^n\Phi^{(b)}\right)\right] . \qquad (37)$$

Presented equations generalize the gravity field equations in the three-parameter model considered in [21, 22, 23].

3 Application of the effective metric formalism

3.1 pp-wave background

Consider now one specific plane-symmetric spacetime associated usually with a gravitational radiation. We assume the metric to be of the form

$$ds^2 = 2dudv - L^2(u)\left[e^{2\beta(u)}(dx^2)^2 + e^{-2\beta(u)}(dx^3)^2\right], \tag{38}$$

where $u = (t - x^1)/\sqrt{2}$ and $v = (t + x^1)/\sqrt{2}$ are the retarded and advanced time, respectively. This spacetime is known to admit the G_5 group of isometries [24], and three Killing four-vectors, ξ^k, $\xi^k_{(2)}$ and $\xi^k_{(3)}$ form three-dimensional Abelian subgroup G_3. The four-vector ξ^k is the null one and covariantly constant, i.e.,

$$\xi^k = \delta^k_v\,, \quad g_{kl}\,\xi^k\xi^l = 0\,, \quad \nabla_l\,\xi^k = 0\,. \tag{39}$$

The four-vectors $\xi^k_{(\alpha)}$ (here and below $\alpha = 2, 3$) are space-like and orthogonal to ξ^k and each others, i.e.,

$$\xi^k_{(\alpha)} = \delta^k_\alpha\,, \quad g_{kl}\,\xi^k_{(2)}\xi^l_{(3)} = 0\,, \quad g_{kl}\,\xi^k\xi^l_{(\alpha)} = 0\,. \tag{40}$$

Two tetrad vectors, $X^k_{(\alpha)}$, are associated with $\xi^k_{(\alpha)}$

$$X^k_{(2)} = \delta^k_2\,\frac{e^{-\beta}}{L}\,, \quad X^k_{(3)} = \delta^k_3\,\frac{e^{\beta}}{L}\,, \quad g_{kl}X^k_{(\alpha)}\;X^l_{(\beta)} = -\delta_{(\alpha)(\beta)}\,. \tag{41}$$

The non-vanishing components of the Ricci and Riemann tensors are, respectively

$$R_{uu} = R^2_{\ u2u} + R^3_{\ u3u}\,, \quad R^2_{\ u2u} = -\left[\frac{L''}{L} + (\beta')^2\right] - \left[2\beta'\frac{L'}{L} + \beta''\right], \tag{42}$$

$$R^3_{\ u3u} = -\left[\frac{L''}{L} + (\beta')^2\right] + \left[2\beta'\frac{L'}{L} + \beta''\right]. \tag{43}$$

We consider a *toy-model*, which satisfies the following requirements. *First*, the background Yang-Mills and Higgs fields are parallel in the group space [25, 26], i.e.,

$$A^{(a)}_k = q^{(a)}A_k\,, \quad \Phi^{(a)} = q^{(a)}\Phi\,, \quad G_{(a)(b)}q^{(a)}q^{(b)} = 1\,, \quad q^{(a)} = const\,. \tag{44}$$

Second, the vector field A_k and scalar field Φ inherit the symmetry of the spacetime, i.e., the Lie derivatives of these quantities along generators of the group G_3, $\{\xi\} \equiv \{\xi^k, \xi^k_{(2)}, \xi^k_{(3)}\}$, are equal to zero:

$$\pounds_{\{\xi\}}A_k = 0\,, \quad \pounds_{\{\xi\}}\Phi = 0\,. \tag{45}$$

Third, the vector field A^k is transverse, i.e.,

$$A^k = -\left[A_{(2)}X^k_{(2)} + A_{(3)}X^k_{(3)}\right], \quad \xi^k A_k = 0\,. \tag{46}$$

Fourth, the background scalar field is massless. *Fifth*, the cosmological constant is absent, $\Lambda = 0$. These five requirements lead to the following simplifications.

(i) The fields A_k and Φ depend on the retarded time u only; there are two non-vanishing components of the field strength tensor

$$F^{(a)ik} = q^{(a)} \left[\left(\xi^i \xi^k_{(2)} - \xi^k \xi^i_{(2)}\right) A'_2(u) + \left(\xi^i \xi^k_{(3)} - \xi^k \xi^i_{(3)}\right) A'_3(u)\right] , \qquad (47)$$

the invariant $F^{(a)}_{ik} F^{ik}_{(a)}$ as well as the terms $\mathcal{R}^{ikmn} F^{(a)}_{mn}$ are equal to zero; there is only one non-vanishing component of the derivative $\nabla_k \Phi$, namely, $\partial_u \Phi$.
(ii) The expression in the right-hand-side of (13) vanishes, $\mathcal{H}^{ik}_{(a)}$ coincides with $F^{ik}_{(a)}$; the equations (13) and (12) are satisfied identically.
(iii) The right-hand-side of the equation (23) vanishes and this equation is satisfied identically.
(iv) All the non-minimal terms $T^{(I)}_{ik}$, ..., $T^{(V)}_{ik}$ (33)-(37) disappear, and the coupling parameters q_1, q_2,..., q_5, being non-vanishing, happen to be hidden, i.e., they do not enter the equations for the gravity field.
After such simplifications the total effective stress-energy tensor in the right-hand-side of (29) is the null one (as it should be, [24]), i.e., this tensor has a structure $\xi_i \xi_k T(u)$. Thus, the non-minimal equations for the gravity field (29)-(37) reduce to one equation

$$\frac{L''}{L} + (\beta')^2 = -\frac{1}{2}\kappa T(u) \,, \qquad (48)$$

where

$$T(u) = \frac{1}{L^2}\left[e^{-2\beta(u)} \left(A'_2\right)^2 (u) + e^{2\beta(u)} \left(A'_3\right)^2 (u)\right] + \left(\Phi'(u)\right)^2 \geq 0 \,, \qquad (49)$$

A_2, A_3, Φ are arbitrary functions of the retarded time u, and the prime denotes the derivative with respect to retarded time. As usual, we can treat $\beta(u)$ as arbitrary function and find $L(u)$ as a solution of (48).

3.2 Structure of the linear response tensor

Taking into account the formula (20) introducing two material tensors: the longitudinal one, $C^{ikmn}_{(\mathrm{long})}$, and the transversal one, $C^{ikmn}_{(\mathrm{trans})}$, we have to focus now on their decomposition for the case of pp-wave background. Mention that if we have a representation of the transversal linear response tensor, related to the coupling constants q_1, q_2 and q_3, the representation of the longitudinal one can be obtained by simple replacement $q_1 \to Q_1 q_1$, $q_2 \to Q_1 q_2$ and $q_3 \to Q_1 q_3$. Thus, below we consider the formulas for $C^{ikmn}_{(\mathrm{trans})}$ only. They are

$$C^{ikmn}_{(\mathrm{trans})} = \frac{1}{2}\left(g^{im}g^{kn} - g^{in}g^{km}\right) + \left(\Pi^{im}\xi^k\xi^n - \Pi^{in}\xi^k\xi^m + \Pi^{kn}\xi^i\xi^m - \Pi^{km}\xi^i\xi^n\right) , \qquad (50)$$

where

$$\Pi^{im} \equiv \frac{1}{2} q_2 \left(R^2{}_{u2u} + R^3{}_{u3u}\right) g^{im} - q_3 \left[R^2{}_{u2u} X^i_{(2)} X^m_{(2)} + R^3{}_{u3u} X^i_{(3)} X^m_{(3)}\right] . \qquad (51)$$

Clearly, the velocity four-vector $U^k = \frac{1}{\sqrt{2}}(\delta^k_u + \delta^k_v)$ is an eigenvector of the tensor Π^{im}, and Π is the corresponding eigenvalue:

$$\Pi^{im} U_m = \Pi U^i \,, \quad \Pi \equiv \frac{1}{2} q_2 \left(R^2{}_{u2u} + R^3{}_{u3u}\right) = \frac{1}{2}\kappa q_2 T \,, \qquad (52)$$

thus, the permittivity tensor, impermeability tensor and cross-effect tensor are, respectively, (we omit here the mark "trans" for simplicity)

$$\varepsilon^{im} = \Delta^{im} + \Pi^{im} + 2\Pi\xi^i\xi^m - \sqrt{2}\Pi(U^i\xi^m + U^m\xi^i)\,, \qquad (53)$$

$$\left(\mu^{-1}\right)^{pq} = \Delta^{pq} - 2\eta^p{}_{ik}\eta^q{}_{mn}\Pi^{im}\xi^k\xi^n\,, \quad \nu^{pm} = \sqrt{2}\,\eta^p{}_{ik}\,\xi^k\,\Pi^{im}\,. \qquad (54)$$

When the observer is at rest, and its velocity four-vector is $U^k = \delta^k_0 = (\delta^k_u + \delta^k_v)/\sqrt{2}$, the non-vanishing permittivity, impermeability and magneto-electric coefficients can be easily calculated:

$$\varepsilon^2_2 = 1 + \Pi^2_2\,, \quad \varepsilon^3_3 = 1 + \Pi^3_3\,, \quad (\mu^{-1})^2_2 = 1 - \Pi^3_3\,, \quad (\mu^{-1})^3_3 = 1 - \Pi^2_2 \quad , \qquad (55)$$

$$\nu^{23} = \frac{1}{L^2}\Pi^3_3\,, \quad \nu^{32} = -\frac{1}{L^2}\Pi^2_2\,, \qquad (56)$$

where

$$\Pi^2_2 \equiv \Pi + q_3 R^2{}_{u2u}\,, \quad \Pi^3_3 \equiv \Pi + q_3 R^3{}_{u3u}\,. \qquad (57)$$

There are two principal cases in this model.

(i) $\Pi^2_2 \neq \Pi^3_3$

For such case $\varepsilon^2_2 \neq \varepsilon^3_3$ and $(\mu^{-1})^2_2 \neq (\mu^{-1})^3_3$, thus the symmetry of the model can not be considered as uniaxial. The interesting particular case here is $q_2 + q_3 = 0$, which relates to $\Pi^2_2 = -\Pi^3_3$. When we reconstruct the effective metrics for the pp-wave model of the non-minimally active vacuum, we can use neither the results obtained by Perlick [17] (since the cross-effects of magneto-electricity are present), nor the explicit results of [18] (since the model is not uniaxial). Below we solve this problem using an alternative way.

(ii) $\Pi^2_2 = \Pi^3_3$

This special case can be treated as uniaxial with antisymmetric tensor of cross effects. It is possible when

$$R^2{}_{u2u} = R^3{}_{u3u} \quad \to 2\beta^{\prime}\frac{L^{\prime}}{L} + \beta^{\prime\prime} = 0 \quad \to \beta^{\prime} = \frac{const}{L^2}\,. \qquad (58)$$

In addition, if $q_2 + q_3 = 0$, one obtains $\Pi^2_2 = \Pi^3_3 = 0$, and the non-minimal interactions do not influence the wave propagation in the vacuum.

3.3 Reconstruction of the effective metrics

Since the spacetime remains active from the point of view of curvature coupling (even if coupling parameters are hidden for the background fields), we can now construct the effective metrics for the *test particles*, treating the non-minimal interactions as a consequence of presence of some effective medium with planar spatial symmetry. This means that we take, first, the material tensor $C^{ikmn}_{(a)(b)}$ and decompose it algebraically with respect to associated metrics. Then we consider the Yang-Mills and Higgs equations in the WKB approximation and obtain color and color-acoustic metrics, respectively.

3.3.1 Associated metrics

Let us introduce three symmetric linearly independent tensor fields

$$h^{im}_{(I)} = g^{im} + 2\Pi\xi^i\xi^m \,, \quad h^{im}_{(II)} = \xi^i\xi^m + 2(\Pi - \Pi_2^2)X^i_{(2)}X^m_{(2)} \,,$$
$$h^{im}_{(III)} = \xi^i\xi^m + 2(\Pi - \Pi_3^3)X^i_{(3)}X^m_{(3)} \,. \tag{59}$$

Using these quantities one can show directly, that

$$2C^{ikmn}_{(\mathrm{trans})} = \left[h^{im}_{(I)}h^{kn}_{(I)} - h^{in}_{(I)}h^{km}_{(I)}\right] + \left[h^{im}_{(II)}h^{kn}_{(II)} - h^{in}_{(II)}h^{km}_{(II)}\right] + \left[h^{im}_{(III)}h^{kn}_{(III)} - h^{in}_{(III)}h^{km}_{(III)}\right] . \tag{60}$$

This formula is a particular case of the multi-metric representation [18]

$$C^{ikmn} = \frac{1}{2\hat{\mu}} \sum_{(\alpha)(\beta)} G_{(\alpha)(\beta)} \left[g^{im(\alpha)} \ g^{kn(\beta)} - g^{in(\alpha)} \ g^{km(\beta)}\right] \tag{61}$$

with $\hat{\mu} = 1$, $G_{(\alpha)(\beta)} = \delta_{(\alpha)(\beta)}$ and $(\alpha), (\beta) = (I), (II), (II)$. Thus, the tensor fields $h^{im}_{(I)}$, $h^{im}_{(II)}$ and $h^{im}_{(III)}$ can be indicated as metrics *associated* with $C^{ikmn}_{(\mathrm{trans})}$. The metrics, associated with $C^{ikmn}_{(\mathrm{long})}$, say, $\tilde{h}^{im}_{(I)}$, $\tilde{h}^{im}_{(II)}$ and $\tilde{h}^{im}_{(III)}$, can be defined analogously with the replacement $q_2 \to Q_1 q_2$ and $q_3 \to Q_1 q_3$. The attempts to decompose $C^{ikmn}_{(\mathrm{long})}$ and $C^{ikmn}_{(\mathrm{trans})}$ by using *two* associated metrics face with mathematical contradiction. Thus, we conclude, that an appropriate internal (associated) sub-space is three-dimensional in the presented case, when vacuum interacting with curvature possesses color cross-effects.

3.3.2 Color metrics

In the WKB-approximation the gauge potentials $A^{(a)}_k$ and the field strength $F^{(a)}_{kl}$ can be extrapolated as follows

$$A^{(a)}_k \to \mathcal{A}^{(a)}_k e^{i\Psi} \,, \quad F^{(a)}_{kl} \to i\left[p_k \mathcal{A}^{(a)}_l - p_l \mathcal{A}^{(a)}_k\right] e^{i\Psi} \,, \quad p_k = \nabla_k \Psi \,. \tag{62}$$

Mention that the nonlinear terms in (4) give the values of the next order in WKB approximation, thus, such a model of gauge field is effectively Abelian. In the leading order approximation the Yang-Mills equations reduce to

$$C^{ikmn}_{(a)(b)} \ p_k \ p_m \ \mathcal{A}^{(b)}_n = 0 \,. \tag{63}$$

Projection (63) onto direction $q^{(a)}$ yields

$$C^{ikmn}_{(\mathrm{long})} \ p_k \ p_m \ \mathcal{A}^{(||)}_n = 0 \,, \quad \mathcal{A}^{(||)}_n \equiv \mathcal{A}^{(b)}_n q_{(b)} \,. \tag{64}$$

Convolution of the equations (63) with $P^{(a)(c)}$ gives

$$C^{ikmn}_{(\mathrm{trans})} \ p_k \ p_m \ \mathcal{A}^{(c)(\perp)}_n = 0 \,, \quad \mathcal{A}^{(c)(\perp)}_n \equiv \mathcal{A}^{(b)}_n P^{(c)}_{(b)} \,. \tag{65}$$

Propagation of longitudinal (with respect to direction pointed by $q^{(a)}$) and transversal color waves, presented by the quantities $\mathcal{A}^{(||)}_n$ and $\mathcal{A}^{(c)(\perp)}_n$, respectively, can be

described by the same method, but the analysis is based on the decomposition of $C^{ikmn}_{(\rm long)}$ or $C^{ikmn}_{(\rm trans)}$, correspondingly. Below we analyse the *transversal* case only, the longitudinal one can be described using the replacement $q_2 \to Q_1 q_2$ and $q_3 \to Q_1 q_3$.

Let us project the equation (65) onto the direction given by ξ^k. This procedure yields the scalar relation

$$\left(\xi^k p_k\right) \left[p^l \mathcal{A}^{(c)(\perp)}_l\right] - \left(p^k p_k\right) \left[\xi^l \mathcal{A}^{(c)(\perp)}_l\right] = 0 \,. \tag{66}$$

As usual, we consider the gauge condition of the Landau-type $\xi^l \mathcal{A}^{(c)(\perp)}_l = 0$ for the Yang-Mills potential four-vector, and obtain immediately, that $p^l \mathcal{A}^{(c)(\perp)}_l = 0$, i.e., the particle four-momentum is orthogonal to the four-vector of the field amplitude. Then we project the equation (65) onto the directions given by $X^k_{(2)}$ and $X^k_{(3)}$ and obtain, respectively,

$$\mathcal{A}^{(c)(\perp)}_{(2)} \left\{g^{ik} + \xi^i \xi^k \left[q_2 \left(R^2_{\ u2u} + R^3_{\ u3u}\right) + 2q_3 R^2_{\ u2u}\right]\right\} p_i p_k = 0 \,, \tag{67}$$

$$\mathcal{A}^{(c)(\perp)}_{(3)} \left\{g^{ik} + \xi^i \xi^k \left[q_2 \left(R^2_{\ u2u} + R^3_{\ u3u}\right) + 2q_3 R^3_{\ u3u}\right]\right\} p_i p_k = 0 \,, \tag{68}$$

where $\mathcal{A}^{(c)(\perp)}_{(2)} \equiv X^m_{(2)} \mathcal{A}^{(c)(\perp)}_m$, etc. Clearly, the color metrics for the waves with the polarization $\mathcal{A}^{(c)(\perp)}_{(2)} \neq 0$, $\mathcal{A}^{(c)(\perp)}_{(3)} = 0$, and vice versa are, respectively

$$g^{ik}_{(2)} = g^{ik} + 2\xi^i \xi^k \Pi^2_2 = g^{ik} + \xi^i \xi^k \left[\varepsilon^2_2 - \frac{1}{\mu^3_3}\right] \,, \tag{69}$$

$$g^{ik}_{(3)} = g^{ik} + 2\xi^i \xi^k \Pi^3_3 = g^{ik} + \xi^i \xi^k \left[\varepsilon^3_3 - \frac{1}{\mu^2_2}\right] \,. \tag{70}$$

When $R^2_{\ u2u} \neq -R^3_{\ u3u}$, i.e., $R_{uu} \neq 0$ and $T(u) \neq 0$, these color metrics can be expressed as a linear combinations of g^{ik} and $h^{ik}_{(I)}$. When the wave with mixed polarization propagates in the described background, it splits into two waves, moving with different phase velocities. This is the effect of the gravitationally induced birefringence, described, first, by Drummond and Hathrell [27] for the case of weak pure gravitational wave, and then investigated in detail for the non-linear case in [28, 7, 8, 9, 29]. The results given by (69) and (70) are generalizations of that ones for the case, when the background spacetime is not empty. In case with empty background spacetime the unique component of the Ricci tensor $R_{uu} = R^2_{\ u2u} + R^3_{\ u3u}$ and, consequently, the quantity Π, were equal to zero, and the equality $\Pi^2_2 = -\Pi^3_3$ took place. Thus, when the spacetime is empty and corresponds to the so-called pure pp-wave gravitational background, one of the waves, say, wave with the polarization along Ox^2, is subluminal, but the second one, with the orthogonal polarization, is superluminal (i.e., the wave phase velocities are more or less than speed of light in pure vacuum). In our case the situation is more sophisticated. Depending on the signs and values of the coupling parameters q_2 and q_3 one can obtain two additional cases: first, both waves are subluminal, second, both wave are superluminal. When $q_2 = -q_3$, clearly, the first case is realized, i.e., one of the waves is subluminal, the other is superluminal.

Given interpretation can be motivated by two ways. The first one is direct and uses the dispersion relations. One can rewrite the dispersion relations following from the equations (67),(68) in terms of frequency $\omega \equiv U^k p_k$ and components of the wave three-vector $X^k_{(a)} p_k$ as follows

$$\omega^2_{(2)} = p^2 - 2p_v^2 \Pi^2_2 \,, \quad \omega^2_{(3)} = p^2 - 2p_v^2 \Pi^3_3 \,, \tag{71}$$

where $p^2 \equiv p_1^2 - g^{\alpha\beta} p_\alpha p_\beta$ is a square of the momentum three-vector. Keeping in mind, that the quantities $\omega_{(2)}/p$ and $\omega_{(3)}/p$ are phase velocities of the waves with polarization along Ox^2 and Ox^3, respectively, we can conclude that positive Π^2_2 (or Π^3_3) characterizes the wave with phase velocity less than one, negative one relates to the superluminal wave. The second interpretation is based on the analysis of the effective line elements

$$ds^2_{(\alpha)} \equiv g_{ik(\alpha)} dx^i dx^k \,. \tag{72}$$

Clearly, the metrics on the plane x^2Ox^3 coincide, i.e.,

$$ds^2_{\perp} \equiv g_{22}(dx^2)^2 + g_{33}(dx^3)^2 = ds^2_{(2)\perp} = ds^2_{(3)\perp} \,, \tag{73}$$

as for the metrics on the cross-section tOx^1

$$ds^2_{||(2)} = 2dudv - 2du^2 \, \Pi^2_2 = dt^2(1 - \Pi^2_2) - (dx^1)^2(1 + \Pi^2_2) + 2\Pi^2_2 \, dtdx^1 \,,$$
$$ds^2_{||(3)} = 2dudv - 2du^2 \, \Pi^3_3 = dt^2(1 - \Pi^3_3) - (dx^1)^2(1 + \Pi^3_3) + 2\Pi^3_3 \, dtdx^1 \,,$$
$$ds^2_{||} = 2dudv = dt^2 - (dx^1)^2 \,, \tag{74}$$

they differ one from another. Let us repeat, that the analysis of the problem for the longitudinal wave characterized by $\mathcal{A}^{(||)}_n$ can be done using the same method and the results can be obtained from described above by the replacement $q_2 \to Q_1 q_2$ and $q_3 \to Q_1 q_3$.

3.3.3 Color-acoustic metric

In the WKB approximation the equations for color massless scalar fields reduce to

$$\left\{ \tilde{g}^{ik}_{(\text{long})} q_{(a)} \left[q_{(b)} \Phi^{(b)} \right] + \tilde{g}^{ik}_{(\text{trans})} \left[P_{(a)(b)} \Phi^{(b)} \right] \right\} p_i p_k = 0 \,. \tag{75}$$

Propagation of the longitudinal scalar field $\Phi^{(||)} \equiv q_{(b)} \Phi^{(b)}$ is thus described by the color-acoustic metric $\tilde{g}^{ik}_{(\text{long})}$, as for transversal scalar fields $\Phi^{(\perp)}_{(a)} \equiv P_{(a)(b)} \Phi^{(b)}$, their propagation is characterized by the metric $\tilde{g}^{ik}_{(\text{trans})}$. For the pp-wave background the transversal color-acoustic metric $\tilde{g}^{ik}_{(\text{trans})} \equiv g^{ik} + \Re^{ik}$ takes the following form

$$\tilde{g}^{ik} = g^{ik} + q_5 \xi^i \xi^k \left(R^2_{\ u2u} + R^3_{\ u3u} \right) = g^{ik} + \kappa q_5 \xi^i \xi^k T(u) \,. \tag{76}$$

As in the case of color metrics, the true metric in the plane x^2Ox^3 coincides with the color-acoustic one. Taking into account that

$$\tilde{ds}^2_{||} = 2dudv - q_5 \kappa T du^2 = dt^2 \left(1 - \frac{1}{2} q_5 \kappa T \right) - (dx^1)^2 \left(1 + \frac{1}{2} q_5 \kappa T \right) + q_5 \kappa T \, dtdx^1 \,, \tag{77}$$

we conclude again that the difference appears in the cross-section tOx^1 only. The longitudinal color-acoustic metric $\tilde{g}^{ik}_{(\text{long})} \equiv g^{ik} + Q_2 \Re^{ik}$ can be obtained from the transversal one by the replacement $q_4 \to Q_2 q_4$ and $q_5 \to Q_2 q_5$, and we omit here the corresponding analysis.

4 Exact solution to the non-minimal Yang - Mills equations in the model of parallel fields

Let us omit now the requirements of the WKB-approximation and consider the Yang-Mills equations in the framework of the model with parallel fields. We can use the obtained color metrics as a hint in the searching for exact solutions to the Yang-Mills equations in the Abelian-type model. Using the multiplicative representation $A^{(a)}_k = q^{(a)} A_k$, we omit below the group index (a). The master equations in this case are linear

$$\nabla_k \left[F^{ik} + \mathcal{R}^{ikmn} F_{mn} \right] = 0 \,. \tag{78}$$

The first hint is to use the Landau-type (algebraic) gauge condition $\xi^k A_k = 0$, i.e., $A_v = 0$. Then the standard Lorentz (differential) gauge condition gives

$$\nabla_k A^k = 0 \quad \rightarrow \quad \partial_v A_u + g^{\alpha\beta} \partial_\alpha A_\beta = 0 \,, \tag{79}$$

where again $\alpha, \beta = 2, 3$. The equations for the components A_2 and A_3 take the following form

$$\left[2\partial_u \partial_v + g^{\alpha\beta} \partial_\alpha \partial_\beta + 2\Pi^2_2 \partial^2_v - 2\beta' \partial_v \right] A_2 = 0 \,, \tag{80}$$

$$\left[2\partial_u \partial_v + g^{\alpha\beta} \partial_\alpha \partial_\beta + 2\Pi^3_3 \partial^2_v + 2\beta' \partial_v \right] A_3 = 0 \,. \tag{81}$$

The exact solutions of these equations are

$$A_2 = e^{\beta} B_2(W_{(2)}) \,, \quad A_3 = e^{-\beta} B_3(W_{(3)}) \,, \tag{82}$$

$$A_u = -\frac{1}{L^2 k_v} \left[e^{-\beta} k_2 B_2(W_{(2)}) + e^{\beta} k_3 B_3(W_{(3)}) \right] \,, \tag{83}$$

where the scalar functions $W_{(2)}$ and $W_{(3)}$, the arguments of arbitrary functions B_2 and B_3, can be treated as phases of the corresponding waves:

$$W_{(2)} = W_{(2)}(0) - \frac{k_\alpha k_\beta}{2k_v} \int_0^u du' g^{\alpha\beta}(u') - k_v \int_0^u du' \Pi^2_2(u') + k_v \; v + k_\alpha x^\alpha \,, \tag{84}$$

$$W_{(3)} = W_{(3)}(0) - \frac{k_\alpha k_\beta}{2k_v} \int_0^u du' g^{\alpha\beta}(u') - k_v \int_0^u du' \Pi^3_3(u') + k_v \; v + k_\alpha x^\alpha \,. \tag{85}$$

Here the constants k_v, k_2 and k_3 are initial values of the wave vector at $u = 0$, when the metric (38) coincides with the Minkowski one. The wave four-vectors $K^i_{(2)}$ and $K^i_{(3)}$, defined as

$$K^i_{(2)} \equiv \nabla^i W_{(2)} = -\delta^i_u \left[\frac{k_\alpha k_\beta}{2k_v} g^{\alpha\beta} + k_v \Pi^2_2 \right] + \delta^i_v k_v + \delta^i_2 k_2 + \delta^i_3 k_3 \,, \tag{86}$$

$$K^i_{(3)} \equiv \nabla^i W_{(3)} = -\delta^i_u \left[\frac{k_\alpha k_\beta}{2k_v} g^{\alpha\beta} + k_v \Pi^3_3 \right] + \delta^i_v k_v + \delta^i_2 k_2 + \delta^i_3 k_3 \,, \tag{87}$$

satisfy the relations

$$g_{im} K^i_{(2)} K^m_{(2)} = -k^2_v \Pi^2_2 \,, \quad g_{im} K^i_{(3)} K^m_{(3)} = -k^2_v \Pi^3_3 \,, \tag{88}$$

$$g_{im}K^i_{(2)}K^m_{(3)} = -k_v^2\left(\Pi^2_2 + \Pi^3_3\right) . \qquad (89)$$

Clearly, the equations (88) relate to (67), (68), when $K^i_{(2)}$ and $K^i_{(3)}$ are identified with p^i. In this sense the color metrics $g^{ik}_{(2)}$ and $g^{ik}_{(3)}$ in (69) and (70) were the hints for the reconstruction of the phase scalars (84) and (85), respectively. As well, the potential four-vector is orthogonal to both $K^i_{(2)}$ and $K^i_{(3)}$:

$$g_{im}K^i_{(2)}A^m = g_{im}K^i_{(3)}A^m = 0 \,. \qquad (90)$$

This exact solution is a generalization of the results [28] for the case, when the pp-wave spacetime is not empty, and the parallel null Yang-Mills and Higgs fields form the spacetime background.

5 Exact solution of the Higgs equation in the model of parallel fields

Consider now the equation

$$\nabla_k\left[\left(g^{kn} + \Re^{kn}\right)\nabla_n\Phi\right] = -m^2\Phi \,, \qquad (91)$$

for the given spacetime background. It can be rewritten as

$$\left[2\partial_u\partial_v + g^{\alpha\beta}\partial_\alpha\partial_\beta + q_5\kappa T(u)\partial^2_v + 2\frac{L'}{L}\partial_v + m^2\right]\Phi = 0 \,, \qquad (92)$$

and the exact solution can be obtained in analogy with the one for the Yang-Mills equations. Indeed, when the mass of the test scalar particle, m, is equal to zero, $m = 0$, the basic solution is

$$\Phi{=}\frac{1}{L}B(W) \,, \quad W{=}W(0){+}\frac{m^2}{2k_v}u{-}\frac{k_\alpha k_\beta}{2k_v}\int_0^u du' g^{\alpha\beta}(u'){-}\frac{1}{2}q_5\kappa k_v\int_0^u du' T(u') \,, \qquad (93)$$

where $B(W)$ is arbitrary function of the phase W. When $m \neq 0$, the exact solution is described by the linear combination of sin and cosin functions

$$\Phi = \frac{1}{L}\left[C_1 \cos W + C_2 \sin W\right] \,, \qquad (94)$$

where C_1 and C_2 are constants of integration. Again, the four-gradient of the phase W, the wave four-vector K^i,

$$K^i \equiv \nabla^i W = \delta^i_u\left[\frac{m^2}{2k_v} - \frac{k_\alpha k_\beta}{2k_v}g^{\alpha\beta} - k_v\frac{1}{2}q_5\kappa T\right] + \delta^i_v k_v + \delta^i_2 k_2 + \delta^i_3 k_3 \,, \qquad (95)$$

satisfies identically the eikonal equation $\tilde{g}^{in}K_iK_n = m^2$, where $\tilde{g}^{in}$ is the color-acoustic metric (76).

6 Discussion

According to (48) and (49) the quantity L''/L is negative, thus, the positive background factor $L(u)$ is a monotonically decreasing function of the retarded time. Starting from $L = 1$ at $u = 0$ this function reaches zero value at some moment $u = u^*$. When L tends to zero, the functions Π^2_2 and Π^3_3 increase infinitely, and when they pass the value $\Pi^2_2 = \Pi^3_3 = \pm 1$, the components of the color metrics take zero values (see (74)). Are such singularities pure coordinate ones or they have some physical interpretation? Different aspects of the singularity, appearing in the space-times, related to the pure gravitational pp-waves and waves in the Einstein-Maxwell theory, were discussed very intensely (see, e.g., [30, 31]). The standard way to eliminate the coordinate singularity is well-known: one should make the coordinate transformation [24]

$$u{=}\tilde{u}\,, \quad v{=}\tilde{v}{-}\frac{1}{2}\left[y^2\left(\frac{L'}{L}{+}\beta'\right){+}z^2\left(\frac{L'}{L}{-}\beta'\right)\right]\,, \quad x^2{=}y\frac{1}{L}e^{-\beta}\,, \quad x^3{=}z\frac{1}{L}e^{\beta}\,. \tag{96}$$

Then the effective line elements (72) take the form

$$ds^2_{(\alpha)} = 2dud\tilde{v} - dy^2 - dz^2 - 2H_{(\alpha)}du^2\,, \tag{97}$$

where

$$H_{(\alpha)} \equiv -\frac{1}{2}\left[y^2 R^2{}_{u2u} + z^2 R^3{}_{u3u}\right] + \Pi^\alpha_\alpha\,, \tag{98}$$

(there is no summation over α). Thus, as in the case of pure gravitational wave [24], $det(g^{ik}_{(\alpha)}) = -1$, and the coordinate singularity is avoided. Nevertheless, another question arises. Let us consider, for instance, an observer the normalized velocity four-vector of which has the t component only

$$V^k = \delta^k_t\left[1 - H_{(\alpha)}\right]^{-\frac{1}{2}}\,, \quad t = \frac{1}{\sqrt{2}}(u+\tilde{v})\,, \quad x = \frac{1}{\sqrt{2}}(\tilde{v}-u)\,. \tag{99}$$

Such a four-vector field can not be prolonged through the surfaces $H_{(\alpha)} = 1$, which are described by two equations ($\alpha = 2, 3$) quadratic in the transverse coordinates y and z

$$y^2 R^2{}_{u2u} + z^2 R^3{}_{u3u} = 2\Pi^\alpha_\alpha - 2\,, \tag{100}$$

where $R^2{}_{u2u}$ and $R^3{}_{u3u}$ are given by (42) and (43). These singular surfaces look like the ones, which appear in the *Analog Gravity* theory [11, 12, 13]. In order to clarify the physical sense of such (dynamic) singularity, let us consider two particular cases of motion of the massless particle in one of the color metrics, say, metric with $\alpha = 2$.

(i) Longitudinal motion

Let the transverse components of the particle momentum four-vector be equal to zero, i.e., $P^y = P^z = 0$. Then one obtains

$$g_{ik(2)}P^iP^k = 0 \quad \to 2P^u\left(P^v - H_{(2)}P^u\right) = 0\,. \tag{101}$$

The first solution, $P^u = 0$, relates to the uniform particle motion in the direction Ox, this motion is not influenced by the pp-wave gravity field. The second solution,

$P^v = H_{(2)} P^u$ relates to the motion in the opposite direction with the longitudinal momentum

$$P^x = \frac{1}{\sqrt{2}} \left(P^v - P^u\right) = \frac{1}{\sqrt{2}} P^u \left(H_{(2)} - 1\right) . \qquad (102)$$

Such a motion is not uniform, in particular, P^x vanishes, when the particle reaches the singular surface $H_{(2)} = 1$ (see (100)). In other words, the particle stops when it contacts with the surface (100) and can not cross it.

(ii) Transversal motion
Let now $P^x = P^z = 0$. Then one obtains

$$P^v = P^u , \quad (P^y)^2 = 2(P^u)^2 \left(1 - H_{(2)}\right) . \qquad (103)$$

Again, the particle stops at the surface $H_{(2)} = 1$, and can not cross it. Thus, there are two singular surfaces, $H_{(2)} = 1$ and $H_{(3)} = 1$, which can be indicated as analogs of the so-called *trapped* surfaces, discussed in [11]. The color massless particles, interacting non-minimally with the pp-wave background, can not cross these trapped surfaces.

Color-acoustic metric (76) is also accompanied by (one) singular trapped surface. To obtain its equation we can formally replace $H_{(\alpha)}$ by the $H_{(s)}$

$$H_{(s)} \equiv \frac{1}{2} \left[(q_5 - y^2) R^2{}_{u2u} + (q_5 - z^2) R^3{}_{u3u} \right] , \qquad (104)$$

in the equation $H_{(s)} = 1$. For the transversal motion of the test massive scalar particle ($P^x = P^z = 0$) the eikonal equation yields

$$\tilde{g}_{ik} P^i P^k = m^2 \quad \to (P^y)^2 = 2(P^u)^2 \left[1 - H_{(s)}\right] - m^2 . \qquad (105)$$

This means that the particle stops at the specific surface $H_{(s)} = 1 - \frac{m^2}{2(P^u)^2}$, which depends on the individual value of P^u. Nevertheless, it is clear that the surface $H_{(s)} = 1$ is unreachable for all massive particles, this last equation can be obtained in the limit $P^u \to \infty$.

In the minimal limit with pure gravitational pp-wave, when $q_1{=}q_2{=}q_3{=}q_4{=}q_5{=}0$ and the Yang-Mills and Higgs fields vanish, three surfaces $H_{(1)} = 1$, $H_{(2)} = 1$ and $H_{(s)} = 1$ coincide and form one trapped surface, describing by equation $y^2 - z^2 = 2L^2/(L^2\beta')'$. Thus, the classical results about the behaviour of the test particles in the field of strong gravitational pp-waves can be reinterpreted in terms of theory Analog Gravity.

7 Conclusions

1. Non-minimal interactions in the pp-wave Einstein-Yang-Mills-Higgs system produce color cross-effects, which are analogous to the magneto-electric effect in electrodynamics. These curvature induced cross-effects result in a new feature: the number of basic associated metrics (three for the longitudinal and three for transversal color models) exceeds the number of color metrics (two and two, respectively).

2. Color and color-acoustic metrics for the pp-wave Einstein-Yang-Mills-Higgs system are presented explicitly, and exact solutions to the Yang-Mills and Higgs equations describing the color and color-acoustic waves are obtained for the model with parallel fields.

3. In the framework of seven-parameter non-minimal Einstein-Yang-Mills-Higgs model with the pp-wave symmetry six trapped surfaces appear: two for the longitudinal color waves, two for the transversal ones and two for color-acoustic waves (longitudinal and transversal). The equations of these surfaces depend on the values of five parameters q_2, q_3, q_5, Q_1 and Q_2, the parameters q_1 and q_4 being hidden, since the Ricci scalar in this model vanishes.

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